

INTRODUCTION TO ERRORS AND ERROR ANALYSIS

To many students and to the public in general, an error is something they have done wrong. However, in science, the word “error” means the “uncertainty” which accompanies every measurement. No measurement of any sort is complete without a consideration of this inherent error. We cannot avoid the “uncertainties” by being very careful. So how do we deal with the measurement errors? All we can do is to try to ensure they are as small as possible and to have a reliable estimate of how large they are.

An important component of a science student’s education is to learn how to handle and interpret experimental data and results. This includes the development of methodologies needed to estimate the errors inherent in various types of measurements, and techniques for testing data to find out if these error estimates are valid, and the understanding of the way errors propagate through calculations made using experimental data. Learning how to handle experimental errors will be very useful also in other Sciences and Engineering. This document is a brief introduction to errors and how you approach them in the laboratory. Please read the entire document, but do not get desperate or disappointed if you do not understand all the details. Different experiments deal with different aspect of errors. Mastering error analysis requires extensive practice and will not happen overnight. Consider this document as a resource on how to handle the particular errors you face in your lab work.

1. PRECISION AND ACCURACY: TWO DIFFERENT TYPES OF ERRORS

There are two main types of errors associated with an experimental result. They are referred to as **precision** and **accuracy**. The precision is usually related to the random error distribution associated with a particular experiment or even with a particular type of experiment (for example, in the experiments where the measured parameters have intrinsically large variations between different samples). The accuracy is related to the existence of systematic errors, for example, the incorrect calibration.

The object of a good experiment is to improve both precision and accuracy. Usually in a given experiment one of these two types of errors is dominant, and the scientist devotes most of his or her efforts towards reducing that one. For example, if you are measuring the length of a carrot in a sample of carrots to determine an average value of the length, the natural random variations within the sample of plants are probably going to be much larger than any possible measurement inaccuracy due to a bad manufacturing of the ruler that you use. In a physics laboratory, the relative effects of the precision and accuracy on the final result usually depend on the particular experiment and the particular apparatus. Some experiments in the introductory physics laboratory have relatively large random errors that require repeating measurements to increase the precision. Other experiments are very precise, but the equipment used has built-in sources of inaccuracy that cannot be eliminated.

2. THREE MAJOR SOURCES OF ERRORS

2.1 READING ERROR

Almost all direct measurements involve reading a scale (ruler, caliper, stopwatch, analog voltmeter, etc.) or a digital display (e.g., digital multimeter or digital clock). Sources of uncertainty depend on the equipment we use. One of the unavoidable sources of errors is a reading error. **Reading Error** refers to the uncertainties caused by the limitations of our measuring equipment and/or our own limitations at the time of measurement (for example, our reaction time while starting or stopping a stopwatch). This does not refer to any mistakes you may make while taking the measurements. Rather it refers to the uncertainty inherent to the instrument and your own ability to minimize this uncertainty. A reading error affects the **precision** of the experiment. The uncertainty associated with the reading of the scale and the need to interpolate between scale markings is relatively easy to estimate. For example, consider the millimeter (mm) markings on a ruler scale. For a person with a normal vision it is reasonable to say that the length could be read to the nearest millimeter at best. Therefore, a reasonable estimate of the uncertainty in this case would be $\Delta l = \pm 0.5$ mm which is half of the smallest division. A rule of thumb for evaluating the reading error on analogue readout is to use half of the smallest division (in case of a meter stick with millimeter divisions it is 0.5 mm), but only the observer can ultimately decide what is his/her limitation in error evaluation. Note that it is wrong to assume that the uncertainty is always half of the smallest division of the scale. For example, for a person with a poor vision the uncertainty while using the same ruler might be greater than one millimeter. If the scale markings are further apart (for example, meter stick with markings 1 cm apart), one might reasonably decide that the length could be read to one-fifth or one-fourth of the smallest division.

There are other sources of uncertainty in direct measurements that can be much more important than uncertainties in the scale or display readings. For example, in measuring the distance between two points, the main problem may be to decide where those two points really are. For example, in an optics experiment it is often necessary to measure the distance between the center of the lens and the position of the focused image. But even a thin lens is usually several millimeters thick, which makes locating the center a difficult task. In addition, the image itself may appear to be focused not at a point, but its location may span a range of several millimeters. Parallax is another significant source of a reading error, where the reading depends on your line of sight.

In the first year laboratory you will often use instruments that have a digital readout. For many digital instruments, you may assume that the reading error is $\pm 1/2$ of the last digit displayed; e.g. if the reading of the timer in the free fall experiment is 682.6 ms, the error can be assumed to be ± 0.05 ms, and you should quote (682.60 \pm 0.05) ms. Experiments on electricity often employ multimeters to measure current and voltage. The usual “ $\pm 1/2$ of the last digit” rule, which assumes an instrument that rounds off perfectly, does not work too well with multimeters. Ideally, you must record the specification of the multimeter used in the experiment. For example, a typical listing on the multimeter (supplied by the manufacturer) will be “accuracy $\pm 0.75\%$ of full scale ± 1 digit”. The first percentage number is what we call the systematic error. It will affect the accuracy of

your readings. It is an estimate of systematic differences between different scales of the multimeter. However it is the random error that determines the precision, and gives you an idea of the scatter that you might expect in your readings. Thus, the “ \pm digit” quoted by the manufacturer might be a better estimate of the random error. Though you should quote the systematic error at the end of your experiment when you are comparing your result with some “standard”, it is better to use ± 1 digit for the random error in each reading. For example, if your reading is 3.48 mA, you should quote (3.48 \pm 0.01) mA. While performing the lab, if in doubt how to record the error in a measurement using a multimeter, consult with your TA.

It is usually difficult or impossible to reduce the inherent reading error in an instrument. In some cases (usually those in which the reading error of the instrument approximates a “random error distribution”) it is possible to reduce the reading error by repeating measurements of exactly the same quantity and averaging them.

2.2 RANDOM ERROR

Random Error refers to the spread in the values of a physical quantity from one measurement of the quantity to the next, caused by random fluctuations in the measured value. For example, in repeating measurements of the time taken for a ball to fall through a given height, the varying initial conditions, random fluctuations in air motion, the variation of your reaction time in starting and stopping a watch, etc., will lead to a significant spread in the times obtained. This type of error also affects the precision of the experiment.

2.2.1 Estimating Uncertainties in Repeating Measurements

For example, while using a stopwatch to measure time intervals, the major uncertainty is usually not from reading the dial, but from our reaction time in starting and stopping the watch. The reaction time is unknown and can vary significantly from person to person and even, for the same person, from measurement to measurement. The solution is to repeat the measurement several times. Most likely you’ll obtain somewhat different values for every one of your measurements. The “correct” measured value lies somewhere between the lowest value and the biggest value (the interval gives a probable range). We assume that the best estimate of the measured value is **the average value**. For example, consider the sequence of four time intervals 2.3, 2.4, 2.5, 2.4 (s). The estimate of time interval is 2.4 s (the arithmetic mean of the four individual measurements), and the probable range is 2.3 to 2.5 s. Note that the probable range means that we are reasonably confident (but not necessarily 100% certain) that the actual quantity lies within this range. Also note that the best estimate, 2.4 s, lies near the midpoint of the estimated range of probable values, 2.3 to 2.5 s. This is true in most measurements.

2.2.2 The mean \bar{x} and the standard deviation σ

Suppose that we want to measure the value of a quantity, x . Let us imagine that we do this by making a total number of n measurements of x , the values of each measurement being denoted by x_i , where i takes on the values from 1 to n . Either because of variability in the quantity x or because of inherent and unavoidable random errors in our measuring procedure, not all values of the individual measurements x_i will be the same. Our best estimate of the “true” value of x is then given by the **average** or **mean** value of x :

$$\bar{x} = \frac{1}{n} \sum_1^n x_i$$

Usually we also require some measure of the spread in the values of x_i . This spread is related to the uncertainty in our estimate of the “correct” value of x . It turns out that the “best” estimate of the spread is given by a quantity σ , called the *standard deviation* of x , given by

$$\sigma = \sqrt{\frac{\sum_1^n (x_i - \bar{x})^2}{n - 1}}$$

Calculating standard deviations can be very tedious. You should practice such a calculation at least a couple of times by hand. Later on, for calculations of standard deviation, you can use a calculator that has a standard deviation function.

2.2.3 The standard error, or the error in the mean, σ_m

σ is associated with the error in each individual measurement x_i . However, what we often really need to know is the error in our best estimate of x , which is the mean (best) value. Clearly this error is less than σ , the error in any individual measurement. If it weren't, we could estimate the “true” value of x as well with one measurement as we could with many. A standard deviation σ of individual measurements divided by the square root of the total number of measurements is called the **standard deviation of the mean**. It is also called the **standard error** and is often denoted by σ_m . The formula (which we do not derive here) for calculating the **standard error (the error in the mean)** is

$$\sigma_m = \frac{\sigma}{\sqrt{n}}$$

where σ is the standard deviation of the x_i 's and n is the number of x_i 's.

Thus, our final answer to the question “what is our true measured value of x ?” is

$$x = \bar{x} \pm \sigma_m = \bar{x} \pm \frac{\sigma}{\sqrt{n}}$$

Note that we could reduce the standard error by taking more measurements. However there is no point in taking more than the number required to reduce the standard error to below the reading error.

Remember that the uncertainty in any measured quantity has the same dimensions as the measured quantity.

3. SYSTEMATIC ERRORS AND THE ROLE OF INSTRUMENT CALIBRATION

Systematic Error refers to an error which is present for every measurement of a given quantity; it may be caused by a bias on the part of the experimenter, a miscalibrated or even faulty measuring instrument, etc. Systematic errors affect the **accuracy** of the experiment.

After evaluating the reading error or the standard error, or both if necessary, we have to make sure that the scale of our measuring instrument is checked against an internationally established measuring standard. Such comparison is called *calibration*. In the real world, we frequently find that our measuring scale is in slight disagreement with the standard. For example, if you inspect such simple tools as rulers, you will find out that no two rulers are exactly the same. It is not uncommon to find a discrepancy of 1 mm or even more among meter sticks. The correct calibration of measuring instruments is obviously of great importance. However, in the first year laboratory, the instruments you will use are usually calibrated by the laboratory staff and ready to use (unless explicit lab instructions tell you otherwise).

In addition to all the errors discussed above, there can be other sources of error that may pass unnoticed: variations in temperature, humidity or air pressure, *etc.* Such disturbances are more or less constant during our measurements (otherwise they would appear as random error when the measurement is repeated) and are generally referred to as the systematic errors. Systematic errors are very difficult to trace since we do not know where to look for them. It is important to learn to notice all the irregularities that could become the sources of systematic errors during our experimental work. Moreover, it is particularly important in data-taking always to record some information about the surrounding physical conditions. Such information may help us later on if we discover a serious discrepancy in our experimental results. As a rule, the place, date and time of measurements, and the type and serial numbers and specifications of the instruments which were used must be recorded.

Estimate all your reading errors while you take your data and write them down with your data. Do the same for all manufacturers' error specifications. These usually cannot be “guessed” later on.

4. QUOTING THE ERRORS: ABSOLUTE AND RELATIVE UNCERTAINTIES

In general, the result of any measurement of physical quantity must include both the value itself (best value) and its error (uncertainty). The result is usually quoted in the form

$$x = x_{best} \pm \Delta x$$

where x_{best} is the best estimate of what we believe is a true value of the physical quantity and Δx is the estimate of **absolute** error (uncertainty). Note that depending on the type of the experiment the prevailing error could be random or reading error. In case the reading error and random error are comparable in value, both should be taken into account and treated as two independent errors. You will learn how to calculate Δx in this case in the “Propagation of Errors” section of this document.

The meaning of the uncertainty Δx is that the true value of x probably lies between $(x_{best} - \Delta x)$ and $(x_{best} + \Delta x)$. It is certainly possible that the correct value lies slightly outside this range. Note that your measurement can be regarded as satisfactory even if the accepted value lies slightly outside the estimated range of the measured value.

Δx indicates the reliability of the measurement, but the quality of the measurement also depends on the value of x_{best} . For example, an uncertainty of 1 cm in a distance of 1 km would indicate an unusually precise measurement, whereas the same uncertainty of 1 cm in a distance of 10 cm would result in a crude estimate. Fractional uncertainty gives us an indication how reliable our experiment is. **Fractional uncertainty** is defined as $\Delta x/x_{best}$ where Δx is the absolute uncertainty. Fractional uncertainty can be also represented in percentile form $(\Delta x/x_{best})_{100\%}$. For example, the length $l = (0.50 \pm 0.01)$ m has a fractional uncertainty of $0.01/0.5 = 0.02$ and a percentage uncertainty of $0.02 \times 100 = 2\%$. Note that the fractional uncertainty is a dimensionless quantity. Fractional uncertainties of about 10% or so are usually characteristic of rather rough measurements. Fractional uncertainties of 1 or 2% indicate fairly accurate measurements. Fractional uncertainties much less than 1% are not easy to achieve, and are rare in an introductory physics laboratory.

5. SIGNIFICANT FIGURES

An uncertainty should not be stated with too much precision. The last significant figure in any stated answer should usually be of the same order of magnitude (in the same decimal position) as the uncertainty. For example, the answer 92.81 s with an uncertainty of 0.3 s should be rounded as (92.8 ± 0.3) s. If the uncertainty is 3 s, then the result is reported as (93 ± 3) s. However, the number of significant figures used in the calculation of the uncertainty should generally be kept with one more significant figure than the appropriate number of significant figures in order to reduce the inaccuracies introduced by rounding off numbers. After the calculations, the final answer should be rounded off to remove this extra figure. You can find much more information on significant digits in a separate

document called “Significant Figures”.

6. PRACTICAL HINTS

So far, we have found two different errors that affect the precision of a directly measured quantity: the reading error and the standard error. Which one is the actual error of precision in the quantity? For practical purposes you can use the following criterion. Take one reading of the quantity to be measured, and make your best estimate of the reading error. Then repeat the measurement a few times. If the spread in the values you obtain is about the same size as the reading error or less, use the reading error. If the spread in values is greater than the reading error, take three or four more, and calculate a standard error and use it as the error. In cases where you have both a reading error and a standard error, choose the larger of the two as “the” error. Be aware that if the dominant source of error is the reading error, taking multiple measurements will not improve the precision.

7. COMMON MISTAKES AND MISCONCEPTIONS WITH ERRORS

7.1 In the introductory physics laboratory, it is almost always meaningless to specify the error to more than two significant digits; often one is enough. It is a mistake to write: $x = (56.7 \pm 0.914606)$ cm, or $x = (56.74057 \pm 0.9)$ cm. Instead, write: $x = (56.7 \pm 0.9)$ cm. You cannot increase either the accuracy or precision by extending the number of digits in your mean value beyond the decimal place occupied by the error. Keep in mind that the error, by its nature, denotes the uncertainty in the last one or two significant digits of the main number and therefore any additional digits obtained from multiplication or division should be rounded off at the meaningful position. So, first calculate your error; round it off to one significant figure; then quote the value of your measurement to the appropriate number of significant figures.

7.2 When quoting errors in a result do not use the flawed logic that “my result is x , the handbook gives a value for this quantity as y , thus the error in my result is $\pm(x - y)$.” Your quoted error should be the result of **your own analysis** of your own experiment whereas $(x - y)$ relates to a comparison of your work to other people's work. $(x - y)$ represents the difference between your result and the accepted value. The discrepancy can be used to characterize the consistency between different sets of measurements, but has nothing to do with the estimate of error in your own experiment.

If a result we produce differs significantly from the accepted value, we then are obligated to explain what has produced the difference. But in quoting our own result, we must provide the error of our own experiment.

8.1 PROPAGATION OF ERRORS

In the majority of experiments the quantity of interest is not measured directly, but must be calculated from other quantities. Such measurements are called indirect. As you know by now, the quantities measured directly are not exact and have errors associated with them. While we calculate the parameter of interest from the directly measured values, it is said that the errors of the direct measurements propagate. This section describes how to

calculate errors in case of indirect measurements.

As an example, consider the following problem. Suppose we have measured the value of a quantity x with an uncertainty, which we denote Δx . In order to test a theoretical formula, suppose that we need to calculate y as function of x ; i.e., $y = f(x)$. We want to know the uncertainty in y due to the uncertainty in the value of x . This is equivalent to asking what will be the variation in y (call it Δy) as x varies from x to $(x + \Delta x)$? Mathematically, this variation is given by $\Delta y = f(x + \Delta x) - f(x)$. The answer comes from the differential calculus: if $y = f(x)$ and Δx is small, then

$$\Delta y \approx \frac{dy}{dx} \Delta x = \frac{df}{dx} \Delta x$$

This argument can be extended for the calculation of quantities that are functions of several different measured quantities. All you will need at this point are the results that you can find below for different types of functions.

Note that we neglect the sign in the differential, since the sign of all errors may take on numerical values which are either + or -.

For practical purposes of error calculations in our experiments you can refer to the following section called "PROPAGATION OF INDEPENDENT ERRORS". You can find more examples in the "EXAMPLES OF ERROR CALCULATIONS" section.

8.2 PROPAGATION OF INDEPENDENT ERRORS

Here Δy and the various Δx 's are either standard deviations, standard errors or reading errors, depending on the circumstances.

Rule 1: If two **mutually independent** quantities are being **added or subtracted**:

$$y = x_1 + x_2 \quad \text{or} \quad y = x_1 - x_2 \quad \text{then}$$

$$\Delta y = \sqrt{(\Delta x_1)^2 + (\Delta x_2)^2}$$

Rule 2: If two **mutually independent** quantities are being **multiplied or divided**:

$$y = x_1 x_2 \quad \text{or} \quad y = x_1 / x_2 \quad \text{then}$$

$$\frac{\Delta y}{y} \approx \sqrt{\left(\frac{\Delta x_1}{x_1}\right)^2 + \left(\frac{\Delta x_2}{x_2}\right)^2}$$

Rule 3: If a quantity is raised to a **power:** $y=x^n$ and if Δx is small then

$$\Delta y \approx nx^{n-1}\Delta x$$

Therefore,

$$\frac{\Delta y}{y} \approx n \frac{\Delta x}{x}$$

8.3 EXAMPLES OF ERROR CALCULATIONS

Example 1: If $y = ax$, where a is a constant, then

$$\Delta y = a\Delta x$$

Example 2: If $y = ax_1 \pm bx_2$, where a and b are constants, then

$$\Delta y \approx \sqrt{a^2(\Delta x_1)^2 + b^2(\Delta x_2)^2}$$

Example 3: If two or more mutually independent values are multiplied or divided, (for example, $y = (x_1x_2)/(x_3x_4)$, $y = x_1/(x_2x_3x_4)$, $y = x_1x_2x_3/x_4$, etc) the resultant error is calculated as follows:

$$\frac{\Delta y}{y} \approx \sqrt{\left(\frac{\Delta x_1}{x_1}\right)^2 + \left(\frac{\Delta x_2}{x_2}\right)^2 + \left(\frac{\Delta x_3}{x_3}\right)^2 + \left(\frac{\Delta x_4}{x_4}\right)^2}$$

Example 4: If $y = \ln(x)$, then $\Delta y \approx \Delta x/x$

Example 5: If $y = \cos(ax)$, then $\Delta y \approx a \sin(ax) \Delta x$

9.1 DATA FITTING TECHNIQUES AND GRAPHICAL ANALYSIS

The quote “A picture is worth a thousand words” is particularly true in the analysis of laboratory data. In many experiments it is very useful to make a rough graph of the data points as they are being taken in order to make sure that the experimental data make sense. Carefully drawn and analyzed graphs are often the best way to extract the values of the unknowns and to confirm or deny the theoretical prediction being tested. Finally graphs are essential in presenting your results in a report. Straight line graphs have a special significance. Their interpretation is often simplest, since all straight lines plotted on an x - y graph have the form

$$y = ax + b$$

where a is the slope of the line (the tangent of the angle between the line and the x -axis), while b is the intercept on the y -axis. Usually x and y depend only on the known or measured quantities while a and b contain the unknowns which you are trying to find.

Though a plot of your data points in their raw form will not generally lie on a straight line, you will find that data can usually be made to fit a straight line by manipulating the equation. If there are two and only two unknowns in your experiment, it is often possible to put the theoretical relationship into the form of a linear equation.

You also need to show the uncertainty in each point of your graph. This is done by drawing the so-called error bars. The error bars are the lines corresponding to the size of the error on either side of the data point. The error bars are drawn vertically for errors in y values and horizontally for errors in x values.

When you are attempting to fit a straight line to some data points by hand, draw a line that encompasses a majority of points and passes as close to all of them as possible, taking into account the errors (indicated by “error bars”) which indicate the uncertainty in the position of the points. This “best” line is a graphical estimate of the “average” of the data; by seeing how far on either side of this “best” line you can draw other acceptable lines. Note that all points need not lie on the line; remember that a scatter of points about a theoretical curve is to be expected in the real world. However, deliberately ignoring a few points without a good reason, just because they spoil a perfect fit to the other points is very bad science. Spread the data nicely across the available area by choosing the scales appropriately on the two axes, which may or may not include the origin at $(0,0)$. It is better to avoid the use of an unnatural subdivision of intervals. Use multiples of 2, 5, or 10 whenever possible (not 3 or 7).

9.2 USING THE COMPUTER TO FIT GRAPHS

The computer provides an objective procedure for giving a best fit of a straight line or a specified function (usually polynomial) to your data. The computers calculate the square of the distance between your data points and the curve it is trying to find. It then adjusts the constants of the polynomial representing the curve until the sum of these squares is a minimum. It gives you the values of the constants found from the procedure.

Although the computer has a data-plotting utility, for the majority of our first year labs you don't have to use the computer. At the initial stages of your lab experience you will learn much more about your data if you plot your graphs and calculate the slopes by hand instead of using computers.

REFERENCES

J.R. Taylor, *An Introduction to Error Analysis* (University Science Books, Mill Valley, California, 1982).